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VIBRATION AND STABILITY ANALYSIS OF COMPRESSED ROCKET VEHICLES

by Rudolf F. Glaser

George C. Marshall Space Flight Center
Huntsville, Ala.

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DEFINITION OF SYMBOLS

Symbol	Definition
A	Constant of integration
A_{si}	Shear area between Stations i and i + 1
$\bar{a}$	Acceleration vector
$\bar{a}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \end{bmatrix}$	Acceleration vector at Station i
$\bar{a}_\eta = \begin{bmatrix} a_{\eta 1} \\ a_{\eta 2} \end{bmatrix}$	Acceleration vector of the center of mass
B	Constant of integration
E	Young's modulus of elasticity
f, f_i	Natural frequencies
G	Shear modulus of elasticity
g	Gravitational acceleration
H_i	x-Component of inertia force acting on the i^{th} mass
I_i	Cross sectional moment of inertia between Stations i and i + 1
I_s	Mass moment of inertia of the vehicle about its C. M.
L	Total length of the vehicle
l	Distance between mass points
l_i	Distance between mass points i and i + 1
$\bar{L}$	Product of transfer matrices
l_{ik}	Element of $\bar{L}$

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
m_i	Mass at Station i
M_a	Total mass of the vehicle
M	Bending moment
M_i	Bending moment on the left side of mass i
M_i^*	Bending moment on the right side of mass i
N	Longitudinal force
N_i	Longitudinal force on the left side of mass i
N_i^*	Longitudinal force on the right side of mass i
P_i	Mass moment of inertia at Station i
Q	Shear
Q_i	Shear on the left side of mass i
Q_i^*	Shear on the right side of mass i
R_i	External (inertia) moment acting on the i^{th} mass
$\bar{r}$	Location vector of a point
r	Radius of gyration of the vehicle
$\bar{S}_i$	Stiffness matrix
T	Thrust
t	Time
$\bar{T}_i$	Inertia matrix
V_i	y-Component of inertia force acting on the i^{th} mass
$\bar{v}$	Velocity vector

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
$\bar{v}_o = \begin{bmatrix} v_{01} \\ v_{02} \end{bmatrix}$	Velocity vector of the body fixed coordinate system origin
(x, y, z)	Body fixed system of coordinates
x_i	x-Coordinate of the Station i
$y\ (x,\ t)$	Deflection curve
$y_i\ (t)$	Deflection at Station i
$\bar{y}^{(k)}$	k^{th} modal column
y_{ki}	i^{th} component of the k^{th} modal column
α	Acceleration of the vehicle's mass center
$\bar{\alpha} = \frac{\alpha}{g}$	
α_k	Refers to dynamic instability
α_s	Refers to a static instability
α_{crit}	Critical acceleration
β_{ij}	Coefficients of the inertia loads at Station i; j = 1, 2, 3, 4
β_i	Dimensionless constant defined by either eq. (27) or by eq. (65)
$\delta_i^{(M)}$	Moment influence coefficient
$\delta_i^{(Q)}$	Shear influence coefficient
δ_i	Element of the matrix $\bar{S}_i$
Δ	Frequency determinant
$\theta(x,\ t)$	Rotation angle of any point of the vehicle axis

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
$\theta_i(t)$	Rotation angle at Station i
η	x-Coordinate of the center of mass
λ	Parameter of the eigenvalue problem
λ_k	k^{th} eigenvalue
ξ	Coordinate defined by eq. (23)
$\sigma_i^{(M)}$	Moment influence coefficient
$\sigma_i^{(Q)}$	Shear influence coefficient
σ_i	Element of $\overline{S}_i$
ϕ	Phase angle
$\Omega = \sqrt{\lambda}$	
$\Omega_i = \sqrt{\lambda_i}$	i^{th} Circular frequency of vibration
ω	Angle velocity of the vehicle about its center of mass
$\overline{\omega}$	Rotation vector of the body fixed coordinate system

VIBRATION AND STABILITY ANALYSIS OF COMPRESSED ROCKET VEHICLES

SUMMARY

Because rocket vehicles are exposed to large axial loads caused by the thrust and the inertia forces of the vehicle masses, the stability problem is important. Problems of this kind are nonconservative. To ensure proper calculation of the critical load the dynamic stability criterion must be applied. Such calculation requires performance of a vibration analysis of the loaded system. The analysis developed in this report is a transfer-matrix method based on ordinary and Timoshenko's beam theory. To simplify the procedure the vehicle was modeled by a lumped-mass system and was assumed to be longitudinally rigid. For convenience, damping was neglected. Of course, damping may play an important role in reducing the critical load imposed. For this reason, damping should be the subject of additional studies. To study the influence of mass and stiffness distribution the analysis has been applied to several characteristic cases. Finally, a Saturn V three-stage vehicle has been investigated. However important the stability problem may be for highly accelerated missiles, this analysis indicates that for modern launch vehicles the acceleration is far removed from critical values.

SECTION I. INTRODUCTION

During powered flight, rocket vehicles are heavily loaded by thrust and inertia forces of the vehicle masses; these loads produce large axial forces. For this reason, the problem of stability is significant.

To determine the critical load of any loaded elastic system, generally three stability criteria are available (Ref. 1).

1. <u>The Dynamic Criterion</u>:

   The critical load is the smallest load which when unbalanced by a sufficient disturbance causes a significant departure from the equilibrium position.

2. <u>The Static (Euler) Criterion</u>:

   The critical load is the smallest load under which a nontrivial equilibrium position exists.

3. The Energy Criterion:

The critical load is the smallest load under which the total potential energy of the system is no longer positive definite.

The Dynamic Criterion follows directly from the definition of a stable equilibrium:

If a small disturbance applied on a system results in small vibrations about the equilibrium position, the position is called stable.

As shown in Reference 1, the Dynamic Criterion is the most general of the three. To obtain the Critical Load of any system, this first criterion may be applied; while the application of the second or third criterion to so-called nonconservative systems may result in failure. The classification into conservative and nonconservative systems is based on the nature of the loads. An elastic system may lose its conservative character since the external loads cannot be derived from a stationary unique potential. In Reference 1 it is shown that, for instance, a cantilever beam with a load of constant amount acting in the tangent of its free end represents such a nonconservative system.

To apply the Dynamic Criterion to the case under consideration, system and equilibrium position must be defined. The loaded system consists of the vehicle and the thrust vector of constant value acting at all times in the vehicle end tangent. To explain the equilibrium position the vehicle may be referred to a body-fixed system of coordinates whose x-axis coincides with the undisturbed vehicle axis. If there is no external disturbance, the vehicle accelerated by the thrust moves straight ahead in the direction of the thrust. All the acting forces, namely the thrust and the inertia forces, lie in the long axis of the body. Hence, no lateral deformation, no motion of the vehicle masses relative to the body-fixed, moving coordinate system will occur. This is the (trivial) equilibrium position. Yet, a sufficiently small disturbance will cause small lateral oscillations about this position. (Longitudinal and torsional oscillations will not be considered here.)

For the same reason as in the above-mentioned case of a cantilever beam with a constant tangential load on its free end (Ref. 1), the problem under consideration is also a nonconservative one. To ensure proper solutions the Dynamic Criterion should be applied.

From the definition of stable equilibrium, application of the Dynamic Criterion implies a vibration analysis of the loaded system. To apply the criterion one must find the smallest load which separates the domain of stability from that of instability. The latter domain is characterized by amplitudes increasing with time but without limit. *)

*) The vibration analysis considered is based on linear theory; its validity is therefore restricted to small vibrations.

As explained in Reference 1, two possibilities exist for cross-over from one domain to the other. Either the crossing point means a nontrivial equilibrium position (Euler's static case) or the instability domain will be reached without occupying such a position at any smaller load. This is the dynamic case. In the first case, all three criteria are applicable; in the second case, only the first is applicable. All conservative problems belong to the first group; but nonconservative problems may belong to the first or second group. Since some special considerations (Ref. 2) are essential for recognition of the behavior of nonconservative systems, application of the first (dynamic) criterion is recommended in all cases.

The above-mentioned vibration analysis of a loaded system consists in the solution of a eigenvalue problem. For any given load, eigenvalues

$$\lambda_i;\ i = 1,\ 2,\ 3,\ \ldots.$$

must be determined which assure solutions of the problem. Each eigenvalue yields a solution representing a mode of vibration of the system. The frequencies of these vibrations are given by

$$f_i = \frac{1}{2\pi}\,\Omega_i = \frac{1}{2\pi}\sqrt{\lambda_i} \tag{1}$$

It is well known that instability is caused by complex frequencies. By plotting eigenvalues or frequencies versus load one recognizes rather easily the character of a stability problem. Considering the case in question, the independent variable may be the thrust force T--or better--the acceleration of the center of gravity:

$$\alpha = \frac{T}{M_a}$$

where M_a is the total vehicle mass. Plotting of the frequencies may start at $\alpha = 0$ with the positive frequencies of the unloaded vehicle. The variations of the frequencies (as α increases) lead to frequency curves which may show one or both of the following two characteristics: coalescence of adjacent frequencies or junction with the x-axis. The first case yields a double frequency f at a certain α_k (Fig. 1) and obviously if

$$\alpha > \alpha_k$$

two of the natural frequencies are conjugate complexes. Hence, this case characterizes a dynamic instability while in the second case (Fig. 2) the junction at $\alpha = \alpha_s$ with the x-axis means $f = 0$ and hence, characterizes the Euler--or static instability (nontrivial equilibrium position). Finally, the analysis results in the critical acceleration

$$\alpha_{crit.} = \min.\ (\alpha_k,\ \alpha_s)$$

A rocket vehicle has, in general, highly nonuniform mass and stiffness distributions. A rocket vehicle must be modeled by an adequate lumped-parameter system that represents as closely as possible the dynamic behavior of the actual structure. Generally, a lumped-mass system will be used. A lumped-mass system has discrete mass points located on the axis of a massless beam having constant stiffness between successive mass points (Fig. 11a). The analysis presented in this report is restricted to such systems; however, extensions to more accurate models are possible without difficulties.

The developed vibration analysis is a transfer-matrix method. Its derivation based on ordinary and Timoshenko's beam theory is presented in Sections IV and V. Because of the mathematical difficulties involved in taking into account a physically reasonable damping assumption, no damping is considered in this analysis. Application of the analysis (Section III) indicates that modern launch vehicles are sufficiently stiff to keep their acceleration of 5 to 6 g's distant from critical values. At most, it may be of some interest to know the frequency variation within this low g-interval. However, as proved in Reference 1, damping plays an important role in the stability of nonconservative systems (see also Reference 3). Damping may reduce the critical load considerably. Hence, this analysis should be considered as a first step only. Inclusion of damping is suggested for further studies.

The external forces acting on the system masses are the inertia forces of the vehicle motion. Because of the vehicle rotation and deformation, these forces are, in general, of a complicated nature. These forces contain components originating from centrifugal, Coriolis, and other accelerations. As shown in the Appendix, several of these components may be neglected under the restriction to small distortions.

SECTION II. SIMPLE LUMPED MASS SYSTEMS

As mentioned before, rocket vehicles during powered flight are loaded by thrust and the inertia forces of the vehicle masses. Therefore, in addition to the stiffness, the mass distribution is a determining factor of the stability. This is valid, in general, for nonconservative systems. In Reference 1 the following statement is proved:

> "In nonconservative systems, under otherwise equal conditions, the critical load depends highly on the mass distribution."

At first, simple lumped-mass systems may serve for studying the influence of mass and stiffness distributions. The investigation of four typical cases are presented below. Mass and stiffness data of these cases are taken as averages from the Saturn V three-stage launch vehicle of Section III.

1. Equal, equidistant mass points on a uniformly stiff beam.

Figure 3 shows distribution of mass and stiffness as well as the acceleration-frequency plot of a five-mass system. Ordinary and Timoshenko's beam theory are applied.

The plot shows that no nontrivial equilibrium position exists. The critical acceleration $\alpha_{crit.}$ is approximately 43.2 g for the ordinary beam and 34.4g for the Timoshenko beam. Mode shape and frequencies at α = 0 and α = 43.1 g (ordinary beam case) are shown by Figure 4. At α = 43.1 g the differences between the two first frequencies and mode shapes are already quite small.

Better distribution of the total beam mass will increase the critical acceleration. Figure 5 shows these critical values of the same uniformly stiff beam having 5, 7, 8, 9, 11, 14, 15, 17, 19, 20, 21, equal, equidistant point masses of the same total amount. Apparently, these values approach the critical acceleration of the uniform beam asymptotically.

2. Top-heavy five-mass system.

Figure 6 shows the acceleration-frequency plot of a five-mass system having the same stiffness as that represented by Figure 3. Also, the stations carry the same amount of mass and mass moment of inertia--but distances between stations have changed. Instead of 4 equal distances l between the point masses, there are now the distances .5 l, .75 l, 1.25 l, 1.50 l.

The a-f plot now shows dynamic and static instabilities. The critical accelerations are $\alpha_{crit.}$ = 22.07g and $\alpha_{crit.}$ = 19.06g for the ordinary and Timoshenko beam cases respectively. These accelerations are now determined by static (Euler's) instabilities. Figure 7 shows the nontrivial equilibrium position of both cases.

3. Tail-heavy five-mass system.

As shown by Figure 8 this system is that of Figure 6, but reoriented by 180°. The a-f plot shows, in general, the same behavior as in case 2. The critical accelerations, however, of both cases (ordinary and Timoshenko beam) are higher.

4. Tail-heavy and tail-stiff five-mass system.

Mass and stiffness data of this system (Fig. 9) show already the characteristics of rocket vehicles. Static and dynamic instabilities are clear from the a-f plot; however, the critical loads of both cases (ordinary and Timoshenko beam) are determined by dynamic instabilities. It is noticeable that these instabilities arise from coalescence of the second and third frequencies.

SECTION III. SATURN V THREE-STAGE VEHICLE

Mass and stiffness data of the Saturn V three-stage vehicle are given in Table 1. Figure 10 shows the α-f plots for the ordinary and Timoshenko beam case. As in the case II 4, the f_1 curves cross the α-axis while the f_2 and f_3 curves coalesce. In the Timoshenko beam case the critical acceleration is dynamic in nature; it is determined by the coalescence of f_2 and f_3. In the ordinary beam case the accelerations of both instabilities are nearly equal.

As already mentioned in Section I, the acceleration of the Saturn V vehicle will not reach 7g. For this reason, the α-f plot is highly theoretical. Of some interest, however, is the variation of frequencies in this low g-range. Table 2 shows the first three frequencies for α = g, 2g, ... 8g. It can be seen that these variations are very small.

There is another interesting fact. Nonconservative vibration problems belong to the nonsélfadjoint eigenvalue problems. As is well known, the eigenfunctions of such eigenvalue problems do not satisfy the orthogonality relations. For lumped-mass systems having n stations, these relations are given by

$$\sum_{i=1}^{n} m_i y_{ji} y_{ki} = 0 \qquad \text{(ordinary beam case)}$$

$$j \neq k$$

$$\sum_{i=1}^{n} (m_i y_{ji} y_{ki} + P_i \theta_{ji} \theta_{ki}) = 0 \qquad \text{(Timoshenko beam case)}$$

where the columns

$$\begin{bmatrix} y_{j1} \\ y_{j2} \\ \cdot \\ \cdot \\ \cdot \\ y_{jn} \end{bmatrix} \qquad \text{(ordinary beam)}$$

$$
\begin{pmatrix} y_{j1} \\ y_{j2} \\ \cdot \\ \cdot \\ \cdot \\ y_{jn} \\ \theta_{j1} \\ \theta_{j2} \\ \cdot \\ \cdot \\ \cdot \\ \theta_{jn} \end{pmatrix} \qquad \text{(Timoshenko beam)}
$$

are the j^{th} eigenvectors, while

$$m_i, \; P_i \qquad i = 1, 2 \ldots . n$$

are mass and mass moment of inertia respectively at the station i.

If $\alpha = \overline{\alpha} g = 0$ the orthogonality relations are exactly satisfied. As α increases, deviations from zero are generated. For the case under discussion Table 3 shows these deviations of the three lowest modes. The values in the first row ($\overline{\alpha} = 0$) represent calculation errors.

SECTION IV. TRANSFER MATRIX METHOD BASED ON ORDINARY BEAM THEORY

Figure 11a shows a rocket vehicle simplified to a lumped-mass system which is referred to a body-fixed system of coordinates (x, y, z) with x positive to the right. The vehicle axis coincides with the x-axis, mass point 1 at the origin. On the right end of the vehicle the thrust T acts in the negative x-direction.

If only the force T is acting (no gravitational forces) the vehicle moves with accelerated, straight-line motion in the direction of the negative x. T and the inertia forces of the beam masses constitute a system of equilibrium, and since all forces act in the vehicle axis, no relative motion between the vehicle masses occurs. This is the equilibrium state. A sufficient small disturbance at t = 0 causes small oscillations of the mass points about the x-axis. The plane of oscillation is assumed to coincide with the x, y plane of the body-fixed system.

Figure 11b shows the system at a time $t \neq 0$. Only the i^{th} mass point is sketched. Its coordinates are x_i, $y_i = y\,(x_i,\, t)$, its mass m_i. The postulates that the axis is fixed in the body (the center of mass of the system remains on the x-axis) and that there is no rotation of the system about an axis perpendicular to the xy-plane, may be expressed by

$$\left.\begin{aligned} &\sum_{i=1}^{n} m_i\, y_i = 0 \\ &\sum_{i=1}^{n} m_i\, x_i y_i = 0 \end{aligned}\right\} \qquad (2)$$

at $t \neq 0$, the i^{th} mass is attacked by the forces:

$$-m_i\, \ddot{y}_i \qquad i = 1,\, 2,\, \ldots\; n \qquad {*)} \qquad (3)$$

$$H_i\,,\; V_i \qquad i = 1,\, 2,\, \ldots\; n \qquad (4)$$

(3) represents the inertia forces of the oscillatory motion while the forces (4) are the inertia forces caused by the accelerated motion of the vehicle. The forces (4), therefore, have to satisfy the following equilibrium conditions.

$$\left.\begin{aligned} &\sum_{i=1}^{n} H_i = T \\ &\sum_{i=1}^{n} V_i = T y_n' \\ &\sum_{i=1}^{n} \left[V_i\,(L - x_i) - H_i\,(y_n - y_i) \right] = 0 \end{aligned}\right\} \qquad (5)$$

---

*) Here and on the succeeding pages, the notations

$$\ddot{y}_i\,(t) = \left.\frac{\partial^2 y\,(x\tau)}{\partial\,\tau^2}\right|_{\substack{x = x_i \\ \tau = t}} \quad ; \qquad y_i'\,(t) = \left.\frac{\partial y\,(x\tau)}{\partial\, x}\right|_{\substack{x = x_i \\ \tau = t}}$$

will be used.

Using the notations:

$$\left.\begin{array}{ll} \eta & \text{x-coordinate of the C. M.} \\ \begin{bmatrix} a_{\eta 1} \\ a_{\eta 2} \end{bmatrix} & \text{acceleration vector of the C. M.} \\ \dot{\omega} & \text{angle acceleration about the C. M.} \end{array}\right\} \qquad (6)$$

the forces (4) are given by

$$\left.\begin{array}{l} H_i = -m_i\, a_{\eta 1} \\ V_i = -m_i\, [a_{\eta 2} + \dot{\omega}\,(x_i - \eta)\,]. \end{array}\right\} \qquad (7)$$

A derivation of equations (7) in detail is given in the Appendix. From Figure 11b, it may be concluded

$$\left.\begin{array}{l} M_a \begin{bmatrix} a_{\eta 1} \\ a_{\eta 2} \end{bmatrix} = -\,T \begin{bmatrix} 1 \\ y'_n \end{bmatrix} \\ I_s\, \dot{\omega} = T\, [y_n - (L-\eta)\, y'_n] \end{array}\right\} \qquad (8)$$

where

$$I_s = \sum_{i=1}^{n} x_i^2\, m_i - \eta^2 M_a$$

From equations (8) it follows

$$\left.\begin{array}{l} a_{\eta 1} = -\dfrac{T}{M_a} = -\,\alpha \\ a_{\eta 2} = -\,\alpha\, y'_n \\ \dot{\omega} = \dfrac{\alpha}{r^2}\,[y_n - (L-\eta)\, y'_n] \end{array}\right\} \qquad (9)$$

where

$$r^2 = \frac{I_s}{M_a} \tag{10}$$

Insertion of the expression (9) into equations (7) yields:

$$\left.\begin{aligned} H_i &= \alpha\, m_i \\ V_i &= -\alpha\, m_i\, \{-y_n' + \frac{1}{r^2}\, [y_n - (L-\eta)\, y_n']\, (x_i - \eta)\} \end{aligned}\right\} \tag{11}$$

Now it is easy to see that H_i, V_i satisfy equations (5).

Using the notations:

$$\left.\begin{aligned} \beta_{i3} &= -\,(x_i - \eta)\,\frac{m_i}{r^2} \\ \beta_{i4} &= [r^2 + (L-\eta)\,(x_i - \eta)]\,\frac{m_i}{r^2} \end{aligned}\right\} \tag{12}$$

the second equation (11) may be written as follows:

$$V_i = \alpha\, \beta_{i3} y_n + \alpha\, \beta_{i4} y_n'. \tag{13}$$

The distortions y_i, y_i' ($i = 1, 2 \ldots n$) are exposed to the same time law

$$\cos\,(\Omega t + \phi) \tag{14}$$

As usual, this time factor will be cancelled. Because no misconception is expected, the used notations may be retained. Then in Figures 11b and 12a $\ddot{y}$ has to be replaced by

$$-\lambda y_i$$

where $\lambda = \Omega^2$ (15)

Then, from Figure 12a and equation (13) the following equilibrium conditions may be concluded

$$\left.\begin{aligned} M_i^* &= M_i \\ Q_i^* &= -\lambda m_i y_i + Q_i - \alpha\beta_{i3}\ddot{y}_n - \alpha\beta_{i4} y'_n . \end{aligned}\right\} \tag{16}$$

The state in a certain point of the beam is known if the distortions, and also internal and external forces acting in this point, are given. Hence, to describe the state, deflection, and slope, moment and shear are necessary for that purpose. Since the shear force depends on y_n and y'_n (see eq. 13) these two quantities must be added to the four others mentioned before. These six quantities may be written as the state vector. On the left side of the i^{th} mass this vector is

$$\bar{y}_i = \begin{bmatrix} y_i \\ y'_i \\ M_i \\ Q_i \\ y_n \\ y'_n \end{bmatrix}$$

Similarly, the state on the right side of station i is given by (Fig. 12a)

$$\bar{y}_i^* = \begin{bmatrix} y_i \\ y'_i \\ M_i^* \\ Q_i^* \\ y_n \\ y'_n \end{bmatrix} \tag{17}$$

As equations (16) show, the two state vectors are linearly related by

$$\bar{y}_i^* = \bar{T}_i \bar{y}_i \tag{18}$$

where $\overline{T}_i$ is the following matrix

$$\overline{T}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\lambda m_i & 0 & 0 & 1 & -\alpha\beta_{i3} & -\alpha\beta_{i4} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

Sometimes $\overline{T}_i$ is called inertia matrix.

An analogous relation exists between the state vectors y_{i+1} and y_i^*: (Fig. 12b)

$$\overline{y}_{i+1} = \overline{S}_i \, \overline{y}_i^* \tag{20}$$

where $\overline{S}_i$ is the stiffness matrix. To determine $\overline{S}_i$ it is necessary at first to solve the differential equation

$$y'' = -\frac{M}{EI_i} \tag{21}$$

with the boundary conditions (see Fig. 12b)

$$\xi = 0; \; y = y_i; \; y' = y_i'; \tag{22}$$

where I_i is the (constant) cross-sectional moment of inertia between stations i and i + 1, E is Young's modulus of elasticity and

$$\xi = x - x_i \tag{23}$$

M is a function of ξ, from Figure 12b it can be seen that

$$M(\xi) = M_i^* + \xi Q_i^* + (y - y_i) N_i^* \tag{24}$$

and $$N_i^* = N_i + H_i = N_i + \alpha m_i. \tag{25}$$

The latter relation follows from the first equation (11).

Insertion of equation (24) into equation (21) yields

$$y'' + \frac{\beta_i^2}{l_i^2} y = - \frac{\beta_i^2}{l_i^2} \left(\frac{M_i^* + \xi Q_i^*}{N_i^*} - y_i \right) \tag{26}$$

with the abbreviation

$$\beta_i^2 = \frac{N_i^* \, l_i^2}{EI_i} \, . \tag{27}$$

The general solution of the differential equation (26) is:

$$y = A \cos \frac{\beta_i \xi}{l_i} + B \sin \frac{\beta_i \xi}{l_i} - \frac{M_i^* + \xi Q_i^*}{N_i^*} + y_i \tag{28}$$

Proof by insertion. Derivation of equation (28) with respect to ξ gives

$$y' = - \frac{\beta_i}{l_i} A \sin \frac{\beta_i \xi}{l_i} + \frac{\beta_i}{l_i} B \cos \frac{\beta_i \xi}{l_i} - \frac{Q_i^*}{N_i^*} \tag{29}$$

Using the boundary conditions (22) it follows from equations (28) and (29)

$$A = \frac{M_i^*}{N_i^*} \tag{30}$$

$$B = \frac{l_i}{\beta_i} \left(y_i' + \frac{Q_i^*}{N_i^*} \right) \tag{31}$$

Setting $x = x_{i+1}$, one obtains from equations (28), (29), (30), and (31)

$$y_{i+1} = y_i + \frac{\sin \beta_i}{\beta_i} l_i y_i' + \frac{\cos \beta_i - 1}{N_i^*} M_i^* + \frac{l_i}{N_i^*} \left(\frac{\sin \beta_i}{\beta_i} - 1 \right) Q_i^* \tag{32}$$

$$y'_{i+1} = y'_i \cos\beta_i - \frac{\sin\beta_i}{\beta_i}\frac{l_i}{EI_i} M_i^* + \frac{\cos\beta_i - 1}{N_i^*} Q_i^*, \tag{33}$$

With the notations:

$$\left.\begin{array}{lll}
\dfrac{\sin\beta_i}{\beta_i} l_i = \delta_i & ; & \cos\beta_i = \sigma_i \\[2ex]
\dfrac{\sigma_i - 1}{N_i^*} = \delta_i^{(M)} & ; & -\dfrac{\delta_i}{EI_i} = \sigma_i^{(M)} \\[2ex]
\dfrac{\delta_i - l_i}{N_i^*} = \delta_i^{(Q)} & ; & \sigma_i^{(Q)} = \delta_i^{(M)}
\end{array}\right\} \tag{34}$$

equations (32) and (33) go over in

$$y_{i+1} = y_i + \delta_i y'_i + \delta_i^{(M)} M_i^* + \delta_i^{(Q)} Q_i^* \tag{35}$$

$$y'_{i+1} = \sigma_i y'_i + \sigma_i^{(M)} M_i^* + \sigma_i^{(Q)} Q_i^*. \tag{36}$$

From equation (24) it follows for $\xi = l_i$

$$M_{i+1} = -N_i^* y_i + M_i^* + l_i Q_i^* + N_i^* y_{i+1}$$

Now, replacement of y_{i+1} by equation (35) and application of equation (34) yields

$$M_{i+1} = N_i^* \delta_i y'_i + \sigma_i M_i^* + \delta_i Q_i^* \tag{37}$$

where δ_i and σ_i are given by equations (34).

Finally, as shown by Figure 12b

$$Q_{i+1} = Q_i^* \tag{38}$$

The equations (35), (36), (37), and (38), and in addition two identities for y_n and y_n' form the linear system (20) whose matrix $\bar{S}_i$ is given by

$$\bar{S}_i = \begin{bmatrix} 1 & \delta_i & \delta_i^{(M)} & \delta_i^{(Q)} & 0 & 0 \\ 0 & \sigma_i & \sigma_i^{(M)} & \sigma_i^{(Q)} & 0 & 0 \\ 0 & N_i^* \delta_i & \sigma_i & \delta_i & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

Repeated application of the operations (18) and (20) yield

$$\bar{y}_n^* = \bar{L}\,\bar{y}_1 \tag{40}$$

where

$$\bar{L} = (l_{ik}) = \bar{T}_n \bar{S}_{n-1} \bar{T}_{n-1} \ldots \bar{S}_1 \bar{T}_1 \tag{41}$$

and

$$\bar{y}_n^* = \begin{bmatrix} y_n \\ y_n' \\ M_n^* \\ Q_n^* \\ \dot{y}_n \\ y_n' \end{bmatrix} \qquad \bar{y}_1 = \begin{bmatrix} y_1 \\ y_1' \\ M_1 \\ Q_1 \\ y_n \\ y_n' \end{bmatrix} \tag{42}$$

It is well known that the boundary conditions may be inserted into these vectors. The left end is free, therefore

$$\left.\begin{aligned} M_1 &= 0 \\ Q_1 &= 0 \end{aligned}\right\} \tag{43}$$

while on the right side, the conditions are

$$\left.\begin{aligned} M_n^* &= 0 \\ Q_n^* &= -T\,y_n' \end{aligned}\right\} \tag{44}$$

Determination of M_n^* and Q_n^* by the external loads (3) and (4) changes equations (44) into

$$\left.\begin{aligned} \sum_{i=1}^{n} (\lambda m_i y_i + V_i)(L - x_i) - H_i (y_n - y_i) &= 0 \\ \sum_{i=1}^{n} (\lambda m_i y_i + V_i) &= T\,y_n' \end{aligned}\right\} \tag{45}$$

Now, application of equations (5) converts equations (45) into

$$\sum_{i=1}^{n} \lambda m_i y_i x_i = 0$$

and

$$\sum_{i=1}^{n} \lambda m_i y_i = 0 \tag{46}$$

Without regard to the factor λ these two equations, however, are the equations (2). Hence, it follows, realization of the conditions (44) ensures that the coordinate system used is body-fixed.

Equations (40), (41), and (42) with the boundary conditions (43) and (44) result in a linear homogeneous system of equations for the unknowns

$$y_1;\ y_1';\ y_n;\ y_n'$$

which may be written as follows:

$$\left.\begin{array}{l} l_{11}\, y_1 + l_{12}\, y_1' + (l_{15} - 1)\; y_n + l_{16}\, y_n' \;=\; 0 \\ l_{21}\, y_1 + l_{22}\, y_1' + l_{25}\, y_n + (l_{26} - 1)\; y_n' \;=\; 0 \\ l_{31}\, y_1 + l_{32}\, y_1' + l_{35}\, y_n + l_{36}\, y_n' \;=\; 0 \\ l_{41}\, y_1 + l_{42}\, y_1' + l_{45}\, y_n + (l_{46} + T)\; y_n' \;=\; 0 \end{array}\right\} \qquad (47)$$

This system is solvable only if

$$\Delta = \begin{vmatrix} l_{11} & l_{12} & l_{15} - 1 & l_{16} \\ l_{21} & l_{22} & l_{25} & l_{26} - 1 \\ \dfrac{l_{31}}{\lambda} & \dfrac{l_{32}}{\lambda} & \dfrac{l_{35}}{\lambda} & \dfrac{l_{36}}{\lambda} \\ \dfrac{l_{41}}{\lambda} & \dfrac{l_{42}}{\lambda} & \dfrac{l_{45}}{\lambda} & \dfrac{l_{46} + \alpha M_a}{\lambda} \end{vmatrix} = 0 \qquad (48)$$

Δ is a polynomial of λ with coefficients depending on α. If λ_k is a root of $\Delta = 0$ then

$$f_k = \frac{\sqrt{\lambda_k}}{2\,\pi}$$

is a natural frequency of the vibrating system. Insertion of λ_k into (47) yields a solution of these equations:

$$y_{k1} = 1;\; y_{k1}';\; y_{kn};\; y_{kn}'$$

Now in view of (43), the state vector $\bar{y}_1$ is known. To obtain the k^{th} mode shape

$$\bar{y}^{(k)} = \begin{bmatrix} y_{k1} \\ y_{k2} \\ \cdot\cdot \\ \cdot \\ \cdot \\ y_{kn} \end{bmatrix} \qquad (49)$$

λ_k has to be inserted in (19) and then the operations (18) and (20), step-by-step applied. This procedure yields all the state vectors whose first components form the above mode shape. Finally multiplication of (49) by

$$\frac{1}{\sqrt{\sum_{i=1}^{n} m_i \, y_{ki}^2}}$$

gives the normalized k^{th} mode shape.

SECTION V. EXTENSION OF THE TRANSFER-MATRIX METHOD TO TIMOSHENKO BEAMS

The transfer-matrix method based on Timoshenko's beam theory is characteristically analogous to the method based on ordinary beam theory. For these reasons, only the changes are discussed in the following pages; while in general, reference is made to the procedure outlined in Section IV.

Timoshenko's theory includes the effects of rotatory inertia and transverse shear deformation on beam bending. This means, on the one hand, the inclusion of external moments caused by the rotatory inertia of the beam masses; on the other hand, it means that the beam deformation is now governed by two coupled differential equations--not one--as in the ordinary beam case.

Considering the lumped-mass system of Figures 11b and 12, the mentioned external moments are

$$-P_i \ddot{\theta}_i; \quad R_i; \qquad i = 1, 2, \ldots n \tag{50}$$

where

θ_i, P_i are rotation angle and mass moment of inertia at station i,

R_i is the external moment contribution at station i caused by the accelerated beam rotation about its center of gravity.

Hence

$$R_i = - \dot{\omega} \, P_i \tag{51}$$

where $\dot{\omega}$ is defined by the third equation (9).

The forces (50) act in addition to the external forces (3) and (4).

The equations governing the beam deformation are

$$\left.\begin{aligned} G A_{si} (y' - \theta) &= Q + N\theta \\ & \qquad\qquad i = 1, 2 \ldots n \\ \theta' &= - \frac{M}{EI_i}, \end{aligned}\right\} {*)} \qquad (52)$$

where

A_{si} is the constant shear area between stations i and i + 1,

G is the shear modulus of elasticity,

θ (x, t) is the rotation angle of the beam element,

Q, M, N are the internal forces of the beam.

The term $N\theta$ of the first equation represents the component of the longitudinal force which contributes to the shear deformation (Fig. 13).

Changeover from Timoshenko's theory to the ordinary beam theory will be arranged by

$$\frac{1}{G A_{si}} = 0; \; P_i = 0; \; i = 1, 2, 3, \ldots n$$

Because of the included rotatory inertia effect, $\theta_i(t) = \theta(x_i t)$ is now equally entitled to y_i. Hence, it is convenient to describe the state using θ_i instead of y_i'. However, the two quantities are related by the first equation (52).

---

*) Notice $y' = \frac{\partial y(xt)}{\partial x}$; $\theta' = \frac{\partial \theta(xt)}{\partial x}$

The state vectors before and after station i (Fig. 12a) are:

$$\bar{y}_i = \begin{bmatrix} y_i \\ \theta_i \\ M_i \\ Q_i \\ y_n \\ y'_n \end{bmatrix} \quad ; \quad \bar{y}_i^* = \begin{bmatrix} y_i \\ \theta_i \\ M_i^* \\ Q_i^* \\ y_n \\ y'_n \end{bmatrix} \tag{53}$$

Since it is assumed that the thrust is, at all times, perpendicular to the end cross section of the beam, no shear deformation at this point can occur. Thus, from the first equation (52) it follows:

$$y'_n = \theta_n \tag{54}$$

Hence, in the state vectors (53), y'_n may be replaced by θ_n.

The k^{th} mode shape is determined by

$$\bar{y}^{(k)} = \begin{bmatrix} y_{k1} \\ y_{k2} \\ \cdot \\ \cdot \\ \cdot \\ y_{kn} \\ \theta_{k1} \\ \theta_{k2} \\ \cdot \\ \cdot \\ \cdot \\ \theta_{kn} \end{bmatrix}$$

The normalization factor is given by

$$\frac{1}{\sqrt{\sum_{i=1}^{n} (m_i\, y_{ki}^2 + P_i\, \theta_{ki}^2)}}$$

The conditions analogous (2) to keep the coordinate system at all times body-fixed are now

$$\left.\begin{aligned} \sum_{i=1}^{n} m_i\, y_i &= 0 \\ \sum_{i=1}^{n} (m_i\, x_i\, y_i + P_i\, \theta_i) &= 0 \end{aligned}\right\} \qquad (55)$$

while the conditions, analogous to (5) are given by

$$\left.\begin{aligned} \sum_{i=1}^{n} H_i &= T \\ \sum_{i=1}^{n} V_i &= T\theta_n \\ \sum_{i=1}^{n} [V_i\,(L - x_i) - R_i - H_i\,(y_n - y_i)] &= 0 \end{aligned}\right\} \qquad (56)$$

Under consideration of the third equation (9) and equation (54) the external moment (51) may be expressed by

$$R_i = -\alpha\, \frac{P_i}{r^2}\, [y_n - (L - \eta)\, \theta_n]\,.$$

where α, r, η, L are defined by (6) equations (9), (10),*) and Figure 11a.

*) Note:

$$I_s = \sum_{i=1}^{n} (x_i^2\, M_i + P_i) - \eta^2 M_a$$

Setting

$$\left.\begin{aligned} \beta_{i1} &= -\frac{P_i}{r^2} \\ \beta_{i2} &= \frac{P_i}{r^2}\,(L-\eta) \end{aligned}\right\} \qquad (57)$$

R_i may be presented as follows:

$$R_i = \alpha\,\beta_{i1}\,y_n + \alpha\,\beta_{i2}\,\theta_n \qquad (58)$$

Now $\ddot{y}_i$ and $\ddot{\theta}_i$ may be replaced by $-\lambda y_i$ and $-\lambda\theta_i$ where λ is defined by equations (14) and (15). Then, from Figure 12a, equations (3), (13), (51), and (58), the equilibrium conditions may be concluded as

$$\left.\begin{aligned} M_i^* &= \lambda\,P_i\,\theta_i + M_i + \alpha\,\beta_{i1}\,y_n + \alpha\,\beta_{i2}\,\theta_n \\ Q_i^* &= -\lambda\,m_i\,y_i + Q_i - \alpha\,\beta_{i3}\,y_n - \alpha\,\beta_{i4}\,\theta_n \end{aligned}\right\} \qquad (59)$$

From (53) and equations (59) the inertia matrix, which is defined by equation (18), follows as

$$\overline{T}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda P_i & 1 & 0 & \alpha\,\beta_{i1} & \alpha\,\beta_{i2} \\ -\lambda m_i & 0 & 0 & 1 & -\alpha\,\beta_{i3} & -\alpha\,\beta_{i4} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (60)$$

To determine the stiffness matrix $\overline{S}_i$ which is defined by equation (20) the linear system of differential equations (52) with the boundary conditions

$$\xi = x - x_i = 0; \quad y = y_i; \quad \theta = \theta_i \qquad (61)$$

shall be solved.

From Figure 12b and the first equation (11) it can be seen that

$$\left.\begin{aligned} M &= M_i^* + \xi\, Q_i^* + (y - y_i)\, N_i^* \\ Q &= Q_i^* \\ N &= N_i^* = N_i + \alpha\, m_i \end{aligned}\right\} \qquad (62)$$

Elimination of θ from equations (52) and from the boundary conditions (61) *) as also application of equations (62) gives:

$$y'' + \frac{\beta_i^2}{l_i^2}\, y = - \frac{\beta_i^2}{l_i^2} \left(\frac{M_i^* + \xi\, Q_i^*}{N_i^*} - y_i \right) \qquad (63)$$

with the boundary conditions:

$$\xi = 0;\; y = y_i;\; y' = \left(1 + \frac{N_i^*}{GA_{si}} \right) \theta_i + \frac{Q_i^*}{GA_{si}} \qquad (64)$$

where

$$\beta_i^2 = \frac{N_i^*\, l_i^2}{EI_i} \left(1 + \frac{N_i^*}{GA_{si}} \right) \qquad (65)$$

The differential equation (63) has the same shape as equation (26). Hence, the solution y and its derivative y' are given formally by equations (28) and (29). Also the integration constant A which follows from the first boundary condition (64) is given by equation (30). However, equation (29) and the second boundary condition (64) result in

$$B = \frac{l_i}{\beta_i} \left(1 + \frac{N_i^*}{GA_{si}} \right) \left(\theta_i + \frac{Q_i^*}{N_i^*} \right) \qquad (66)$$

*) Where the time factor is already removed.

Setting $\xi = x_{i+1} - x_i = l_i$ from equations (28), (29), (30), (52), (65), and (66) one obtains finally

$$y_{i+1} = y_i + \left(1 + \frac{N_i^*}{GA_{si}}\right) \frac{\sin \beta_i}{\beta_i} l_i \theta_i + \frac{\cos \beta_i - 1}{N_i^*} M_i^*$$

$$+ \frac{l_i}{N_i^*} \left[\left(1 + \frac{N_i^*}{GA_{si}}\right) \frac{\sin \beta_i}{\beta_i} - 1\right] Q_i^* \tag{67}$$

and

$$\theta_{i+1} = \theta_i \cos \beta_i - \frac{\sin \beta_i}{\beta_i} \frac{l_i}{EI_i} M_i^* + \frac{\cos \beta_i - 1}{N_i^*} Q_i^* \tag{68}$$

With the notations:

$$\left.\begin{aligned}
&\left(1 + \frac{N_i^*}{GA_{si}}\right) \frac{\sin \beta_i}{\beta_i} l_i = \delta_i \quad ; \qquad && \cos \beta_i = \sigma_i \\
&\frac{\sigma_i - 1}{N_i^*} = \delta_i^{(M)} \quad ; && \frac{-\delta_i}{\left(1 + \frac{N_i^*}{GA_{si}}\right) EI_i} = \sigma_i^{(M)} \\
&\frac{\delta_i - l_i}{N_i^*} = \delta_i^{(Q)} \quad ; && \delta_i^{(M)} = \sigma_i^{(Q)}
\end{aligned}\right\} \tag{69}$$

equations (67) and (68) go over in

$$y_{i+1} = y_i + \delta_i \theta_i + \delta_i^{(M)} M_i^* + \delta_i^{(Q)} Q_i^* \tag{70}$$

and

$$\theta_{i+1} = \sigma_i \theta_i + \sigma_i^{(M)} M_i^* + \sigma_i^{(Q)} Q_i^* \tag{71}$$

The equations concerning moment and shear may be obtained by the first two equations (62) setting $\xi = x_{i+1} - x_i = l_i$

$$\left.\begin{aligned} M_{i+1} &= M_i^* + l_i Q_i^* + (y_{i+1} - y_i) N_i^* \\ Q_{i+1} &= Q_i^* \end{aligned}\right\} \qquad (72)$$

Now, replacement of $y_{i+1} - y_i$ by equation (70) and application of equations (69) yields:

$$M_{i+1} = N_i^* \delta_i \theta_i + \sigma_i M_i^* + \delta_i Q_i^* \qquad (73)$$

where N_i^*, δ_i, σ_i are given by the third equation (62) and equations (69) respectively.

Equations (70), (71), (72), and (73), and two identities for y_n and θ_n, form a linear system whose matrix is the stiffness matrix $\bar{S}_i$ as defined by equation (20). $\bar{S}_i$ agrees formally with the matrix (39), however, its elements δ_i, $\delta_i^{(M)}$, $\delta_i^{(Q)}$, σ_i, $\sigma_i^{(M)}$, $\sigma_i^{(Q)}$ are now given by equations (65) and (69).

The following considerations are the same as those in the ordinary beam case. Application of the boundary conditions leads to the eigen-frequency determinant, and finally, to the determination of the mode shapes.

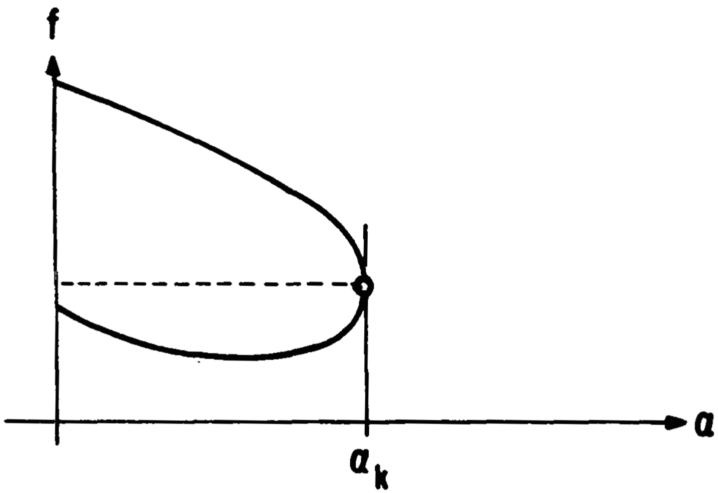


FIGURE 1. COALESCENCE OF TWO ADJACENT FREQUENCIES CHARACTERIZES A DYNAMIC INSTABILITY

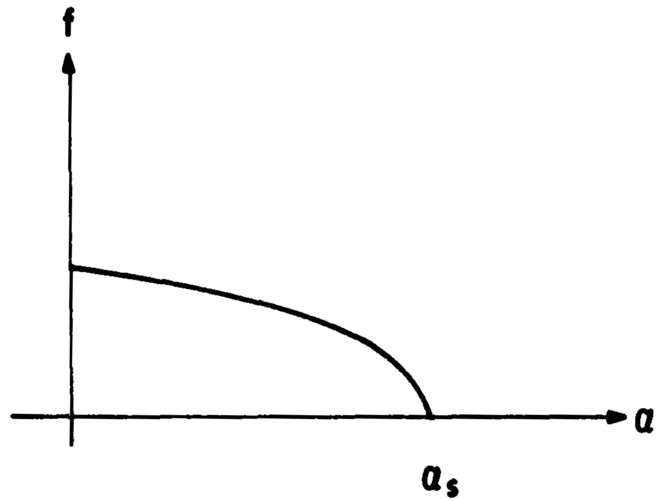
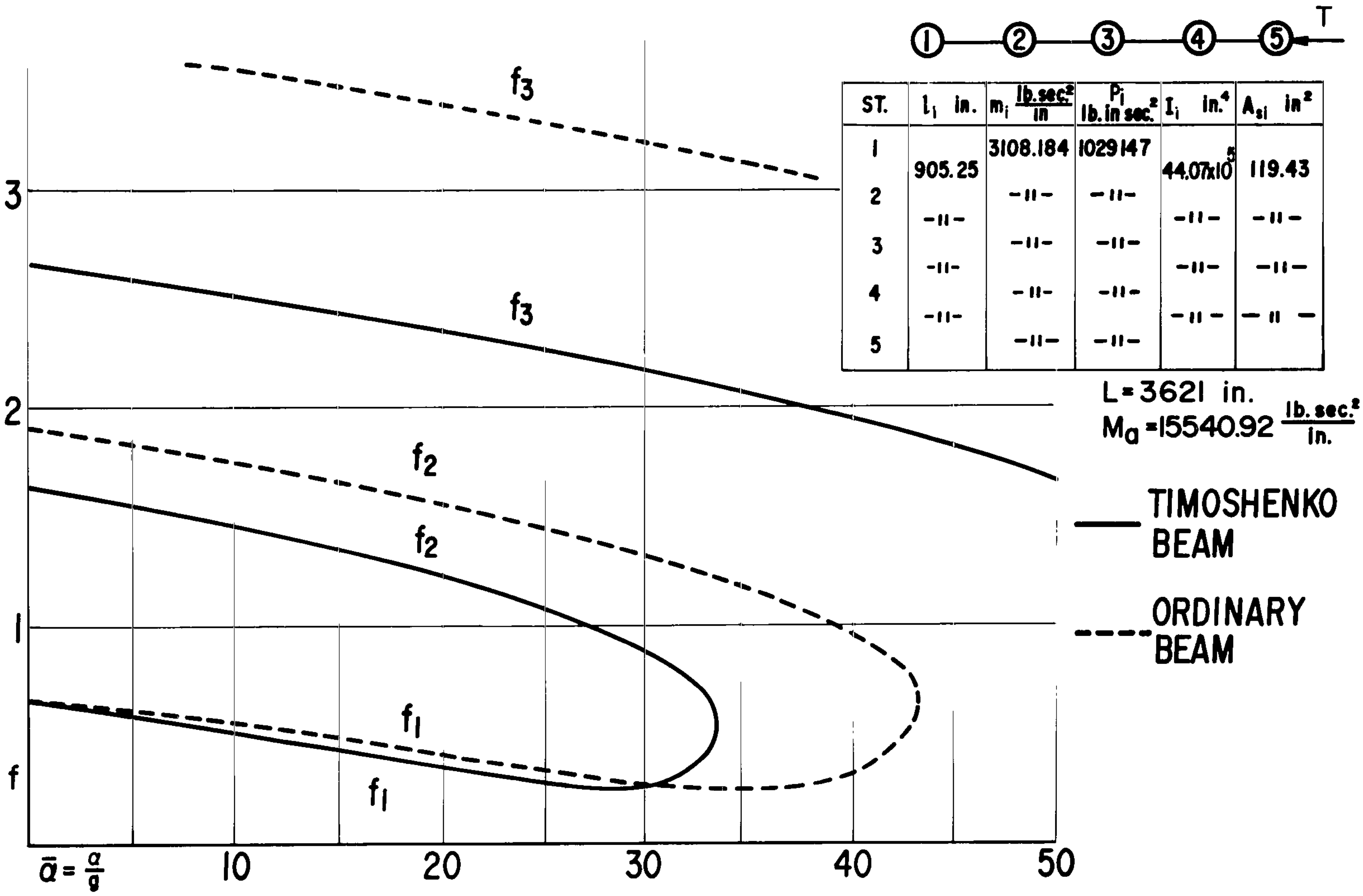


FIGURE 2. JUNCTION OF THE LOAD AXIS CHARACTERIZES A STATIC (EULER'S) INSTABILITY



ST.	l_i in.	m_i $\frac{\text{lb.sec.}^2}{\text{in}}$	p_i lb. in sec.^{2}	I_i in.^{4}	A_{si} in^2
1		3108.184	1029147		
	905.25			44.07×10^5	119.43
2		-"-	-"-		
	-"-			-"-	-"-
3		-"-	-"-		
	-"-			-"-	-"-
4		-"-	-"-		
	-"-			-"-	-"-
5		-"-	-"-		

FIGURE 3. ACCELERATION-FREQUENCY PLOT OF A UNIFORMLY STIFF BEAM CARRYING FIVE EQUAL EQUIDISTANT MASS POINTS

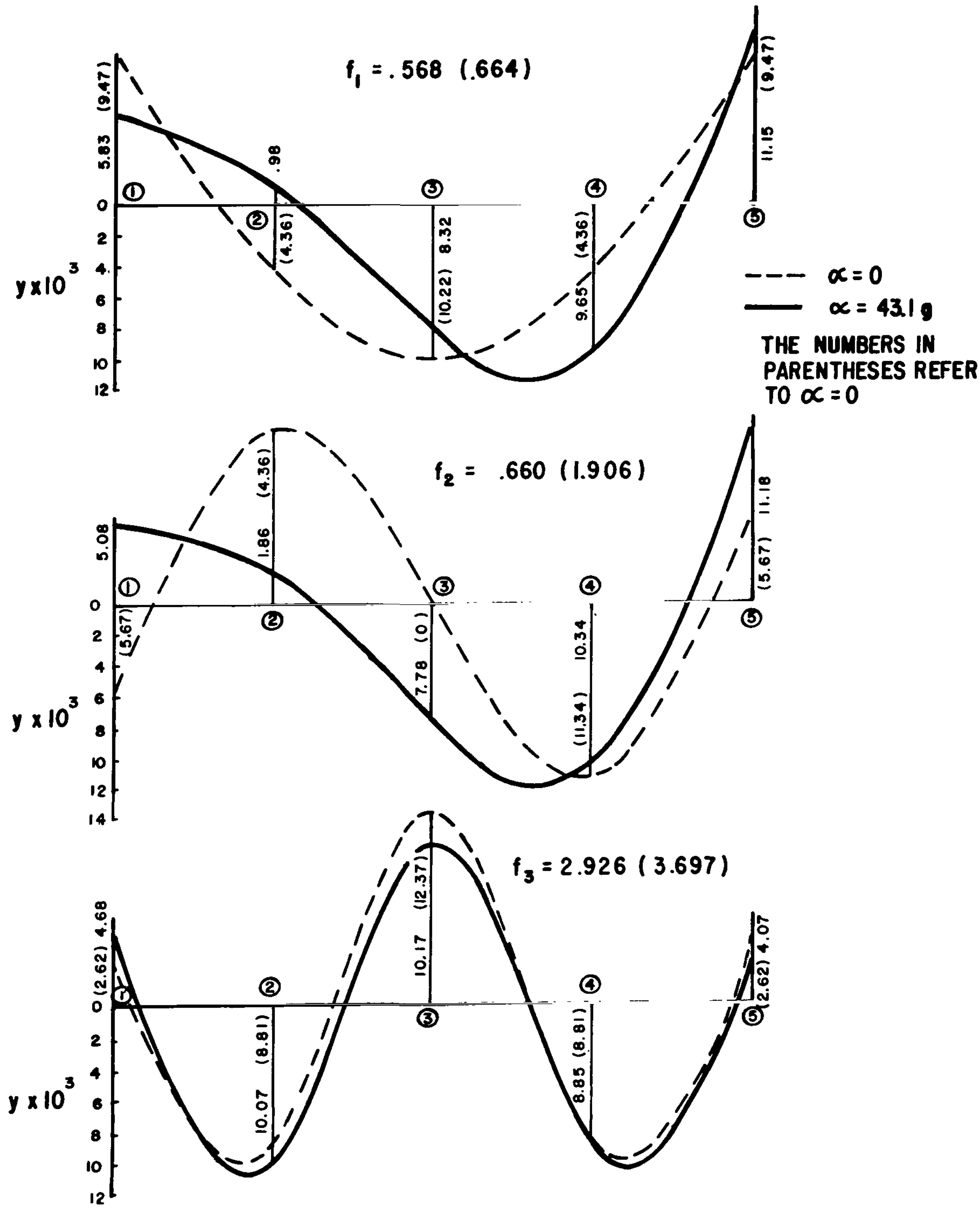


FIGURE 4. THE FIRST THREE FREQUENCIES AND MODE SHAPES (NORMALIZED) OF THE FIVE MASS SYSTEM SHOWN BY FIG. 3 AT $\alpha = 0$ AND $\alpha = 43.1$ g (ORDINARY BEAM CASE)

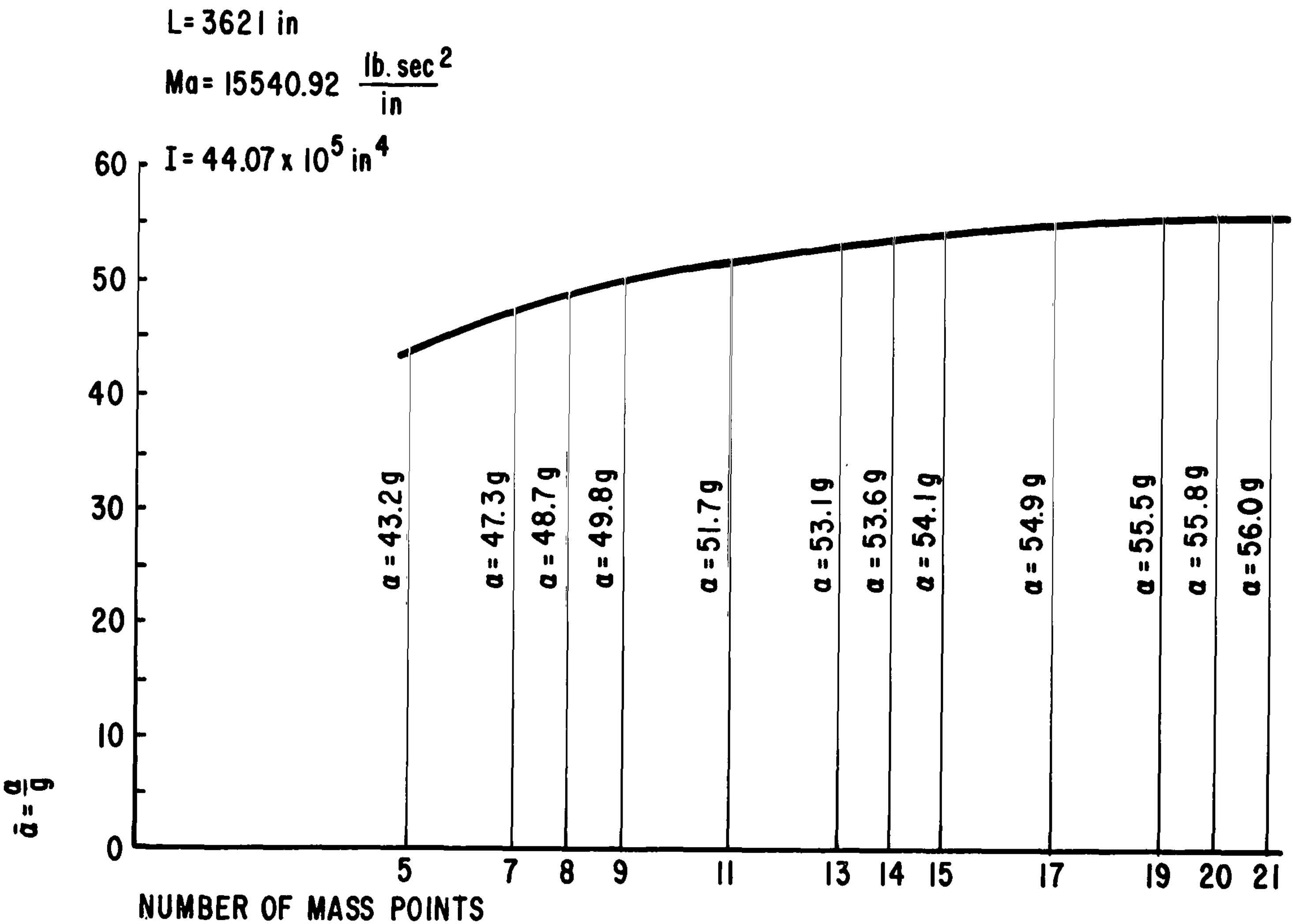
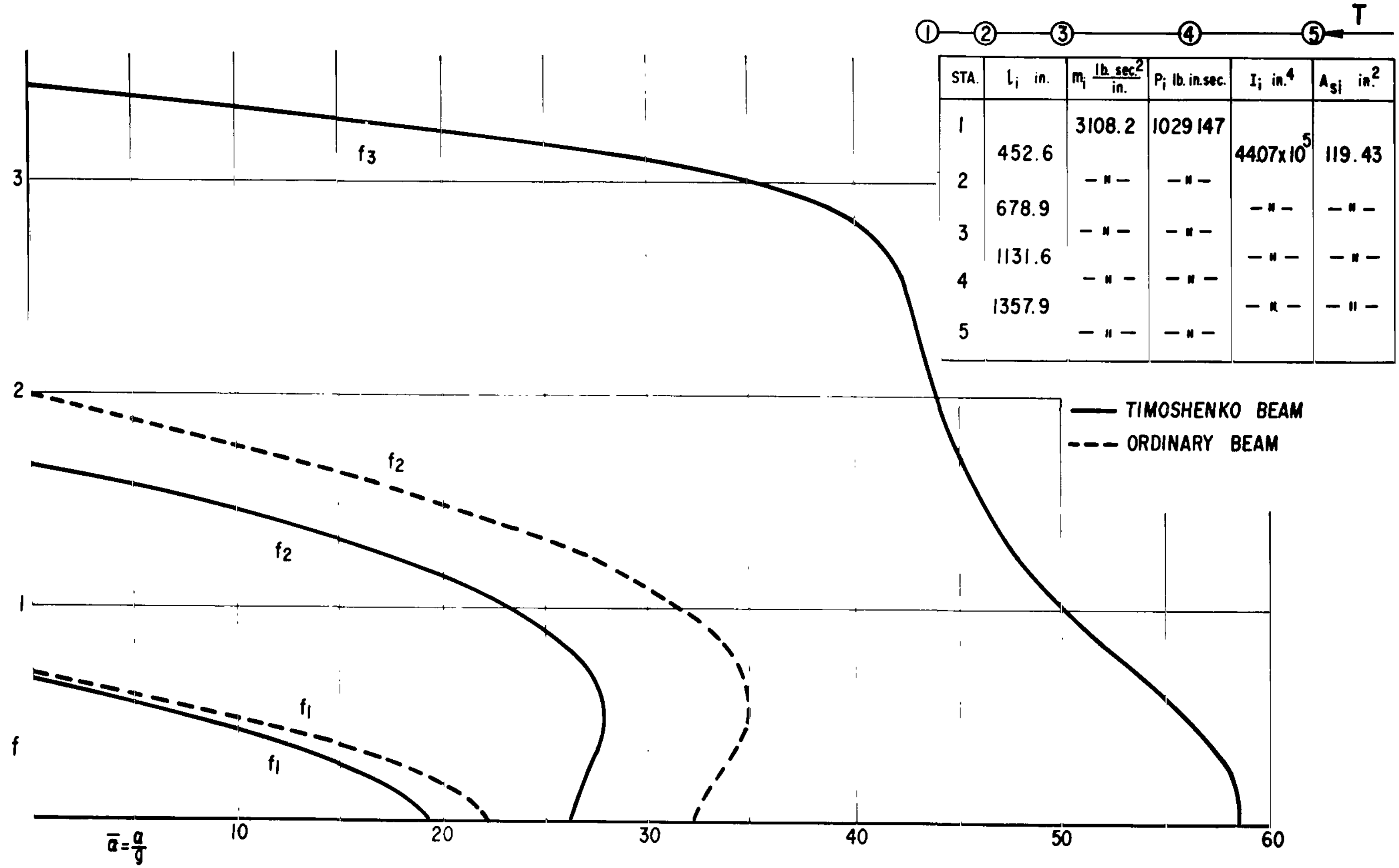


FIGURE 5. CRITICAL ACCELERATIONS OF A UNIFORMLY STIFF BEAM CARRYING EQUAL EQUIDISTANT POINT MASSES OF CONSTANT TOTAL AMOUNT (ORDINARY BEAM)



STA.	l_i in.	m_i $\frac{\text{lb. sec}^2}{\text{in.}}$	P_i lb. in.sec.	I_i in.^{4}	A_{si} in.^{2}
1		3108.2	1029147		
	452.6			4407×10^5	119.43
2		— " —	— " —		
	678.9			— " —	— " —
3		— " —	— " —		
	1131.6			— " —	— " —
4		— " —	— " —		
	1357.9			— " —	— " —
5		— " —	— " —		

FIGURE 6. ACCELERATION-FREQUENCY PLOT OF A TOP-HEAVY FIVE MASS SYSTEM

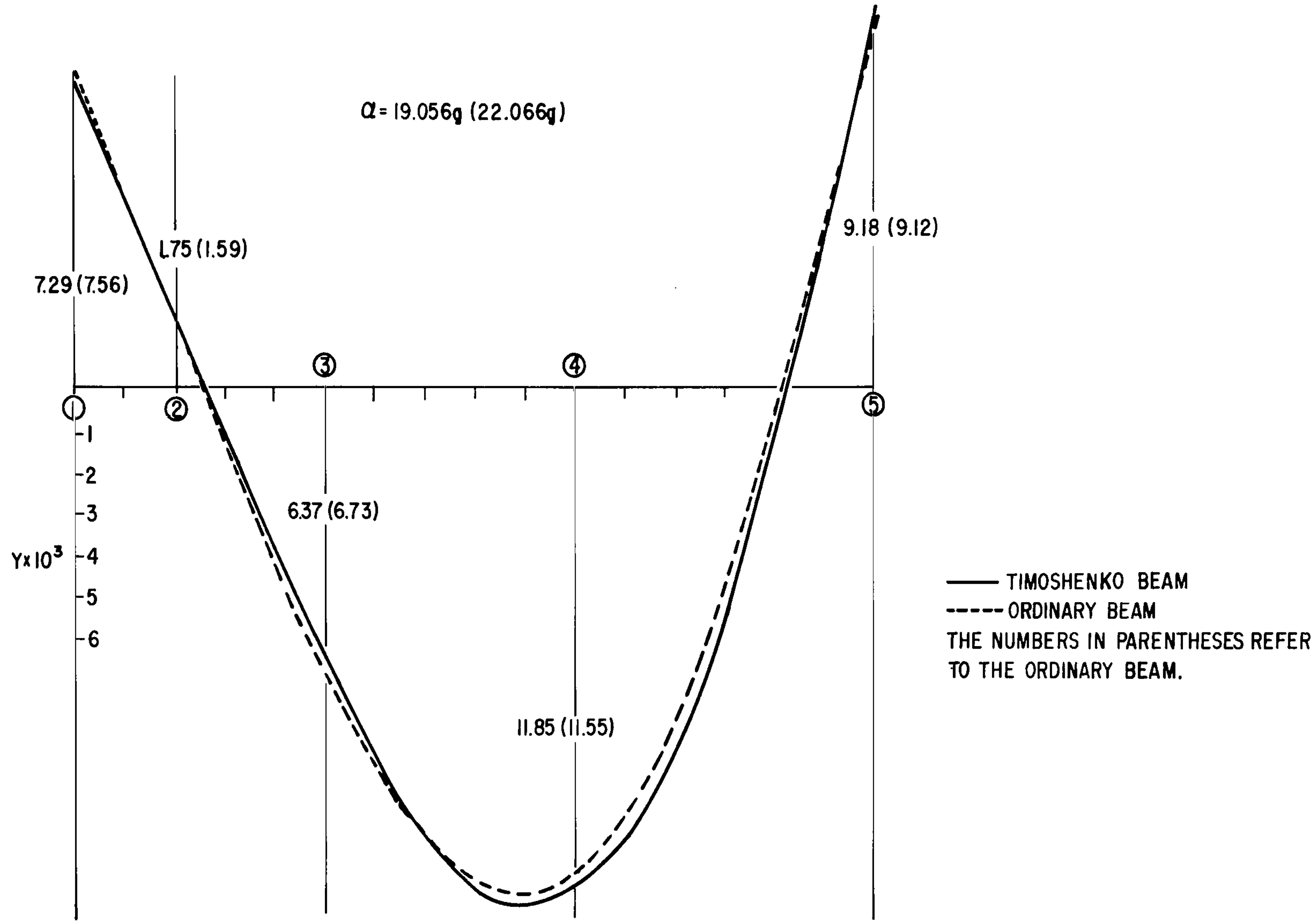
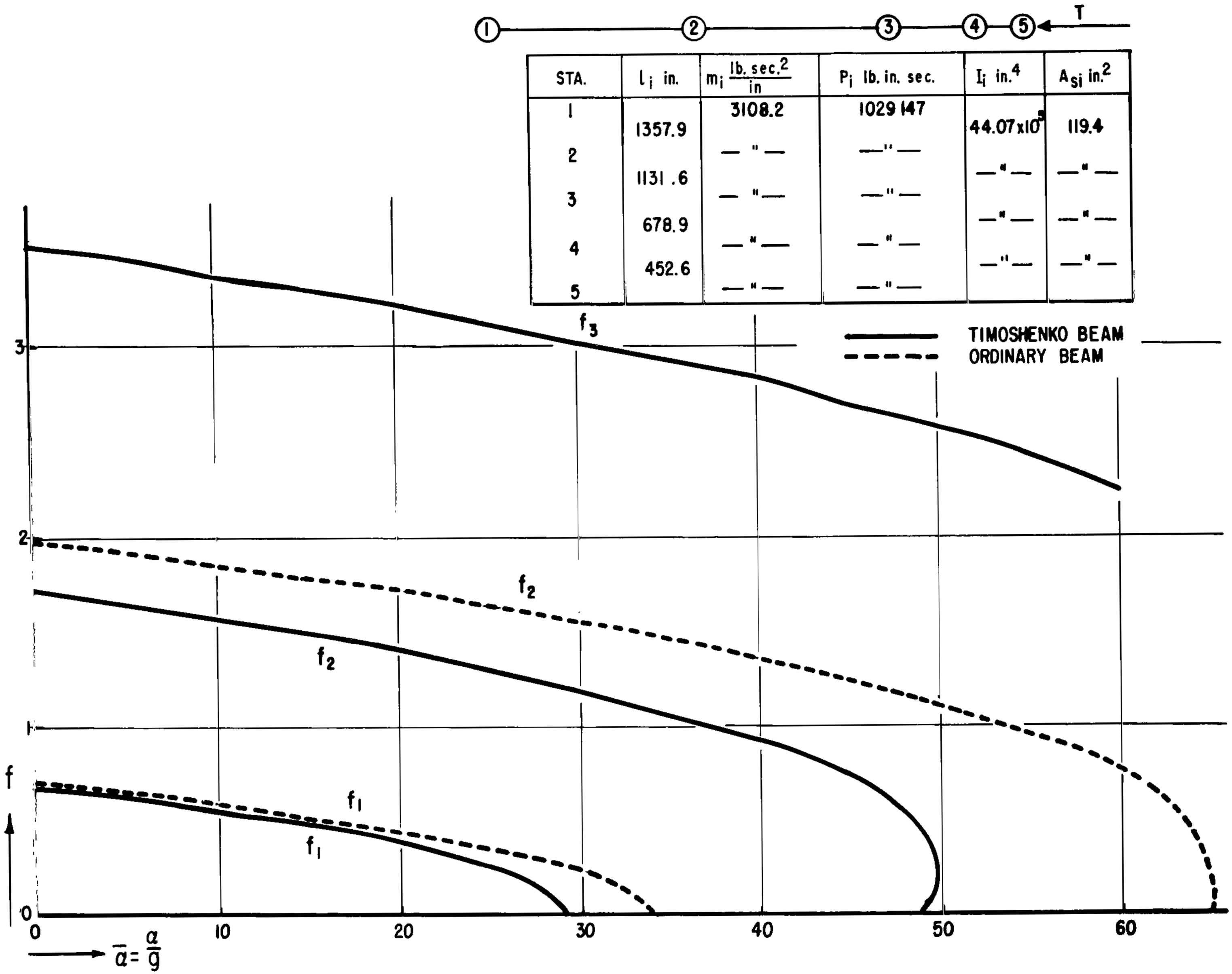
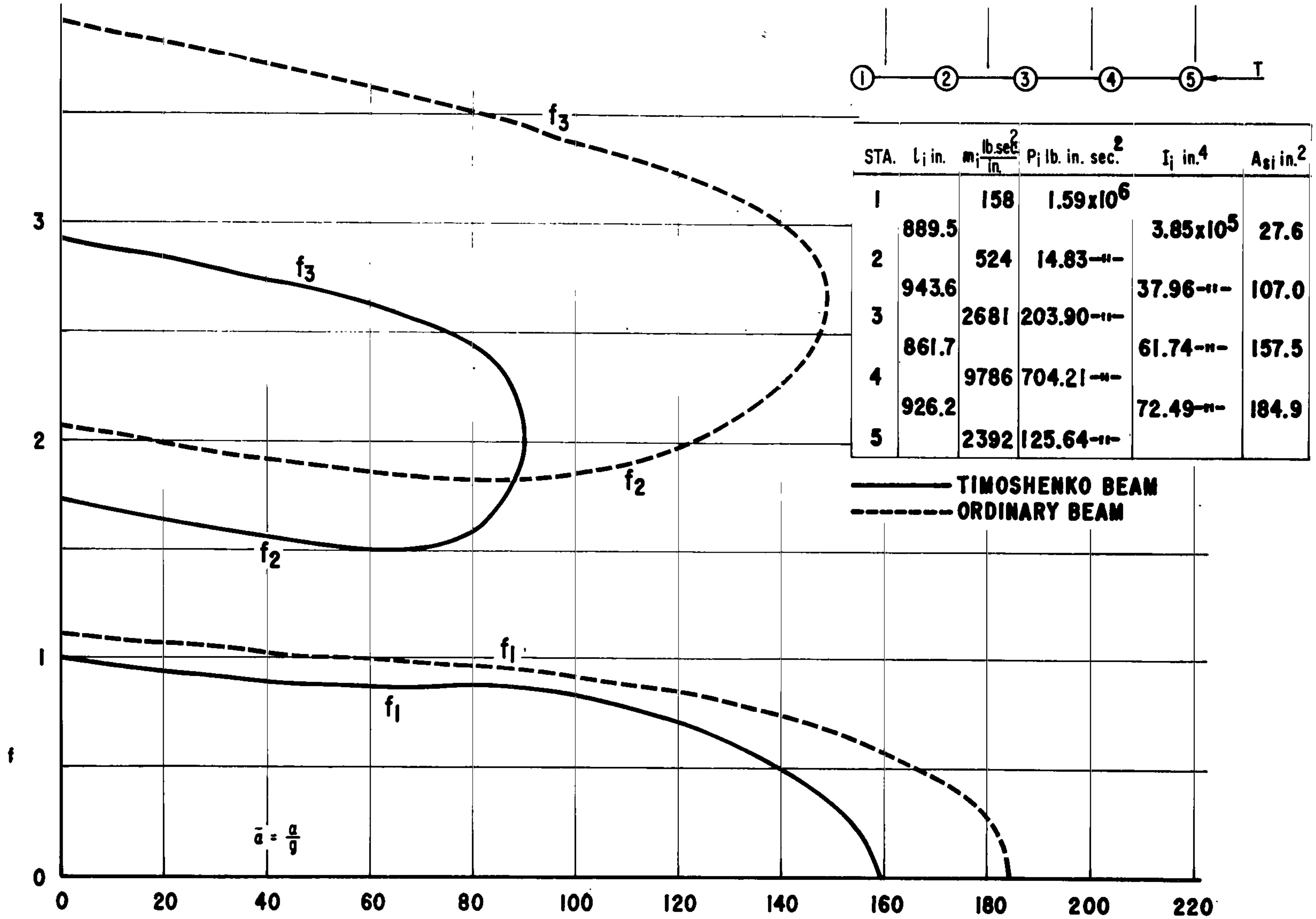


FIGURE 7. NONTRIVIAL EQUILIBRIUM POSITION (NORMALIZED) OF THE TOP-HEAVY 5 MASS SYSTEM SHOWN BY FIGURE 6



STA.	l_i in.	m_i $\frac{\text{lb. sec.}^2}{\text{in}}$	P_i lb. in. sec.	I_i in.^{4}	A_{si} in.^{2}
1		3108.2	1029 147		
	1357.9			44.07×10^5	119.4
2		— " —	— " —		
	1131.6			— " —	— " —
3		— " —	— " —		
	678.9			— " —	— " —
4		— " —	— " —		
	452.6			— " —	— " —
5		— " —	— " —		

FIGURE 8. ACCELERATION-FREQUENCY PLOT OF A TAIL-HEAVY FIVE MASS SYSTEM



STA.	l_i in.	$m_i \frac{lb.sec^2}{in.}$	P_i lb. in. sec.^{2}	I_i in.^{4}	A_{si} in.^{2}
1		158	1.59×10^6		
	889.5			3.85×10^5	27.6
2		524	14.83-"-		
	943.6			37.96-"-	107.0
3		2681	203.90-"-		
	861.7			61.74-"-	157.5
4		9786	704.21-"-		
	926.2			72.49-"-	184.9
5		2392	125.64-"-		

FIGURE 9. ACCELERATION-FREQUENCY PLOT OF A TAIL-HEAVY AND TAIL-STIFF FIVE MASS SYSTEM

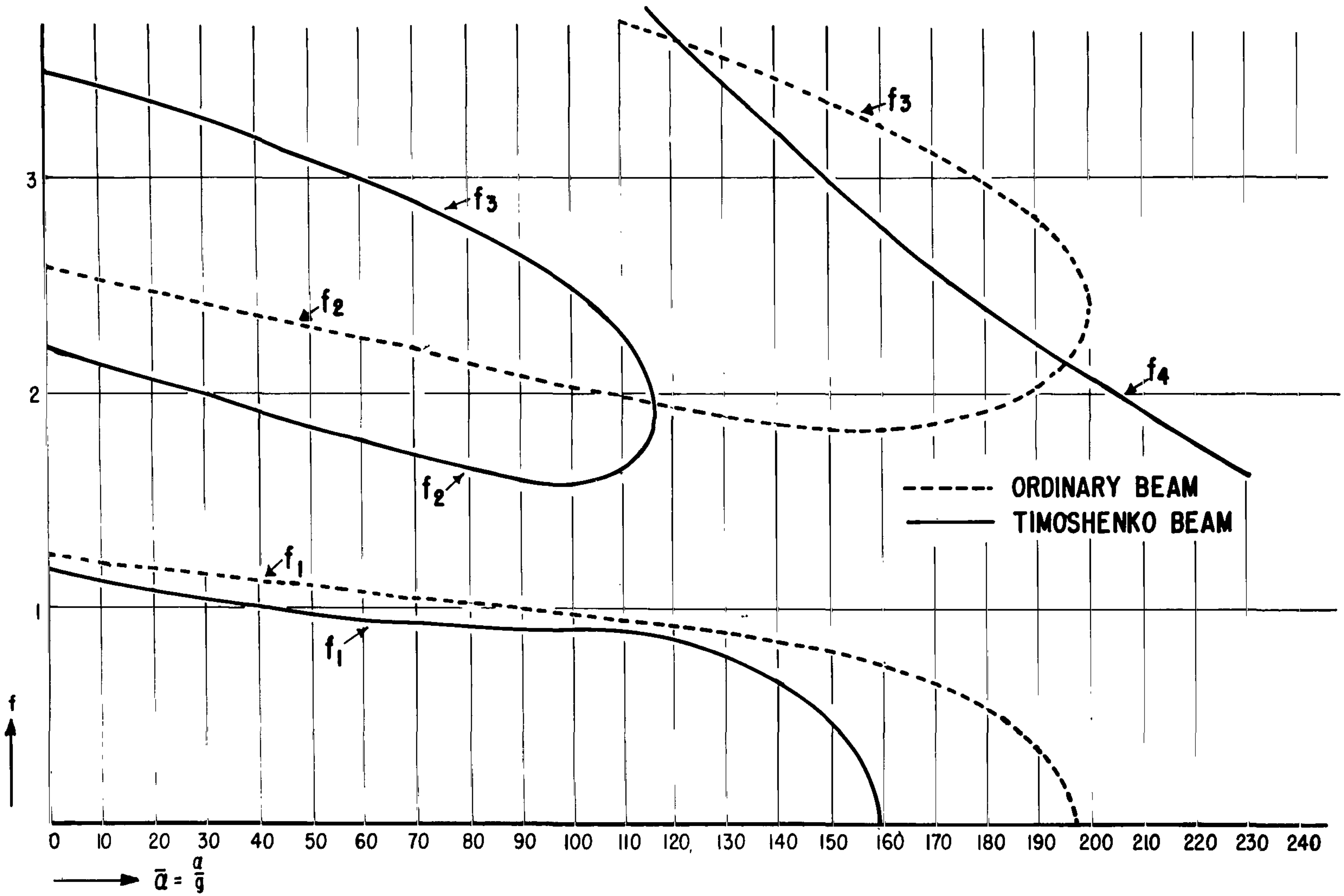
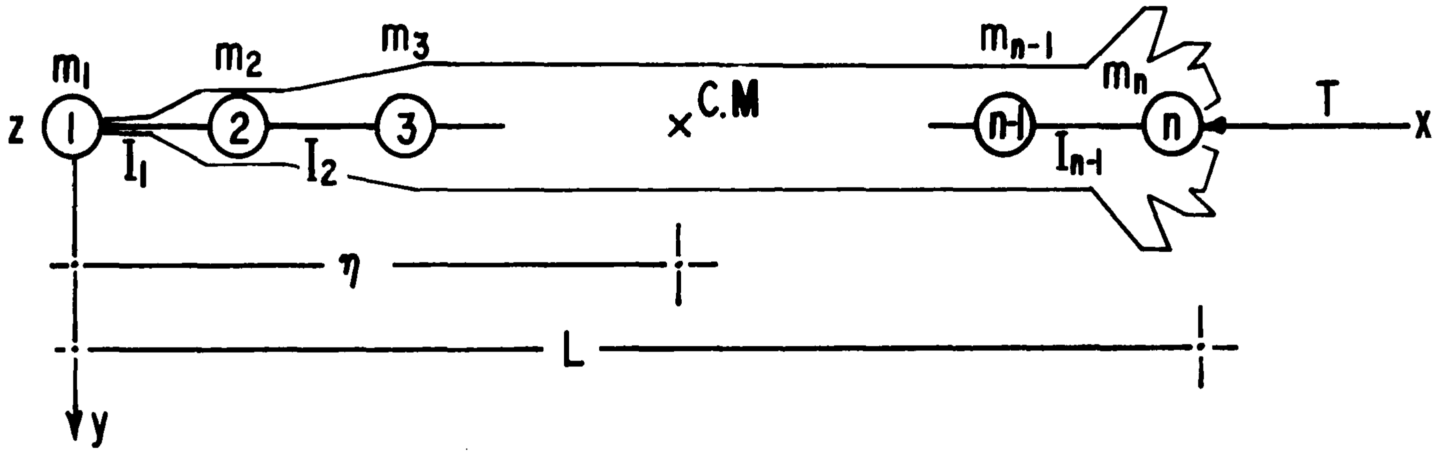
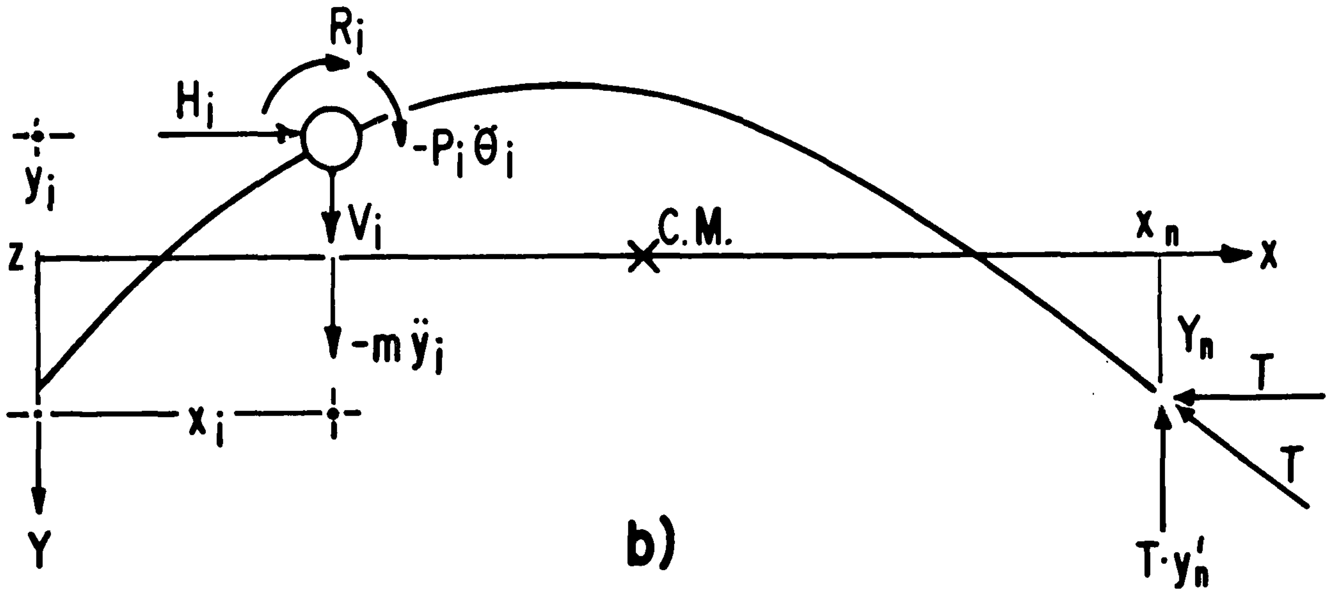


FIGURE 10. SATURN V 3 STAGE LAUNCH VEHICLE; α - f PLOT



a)

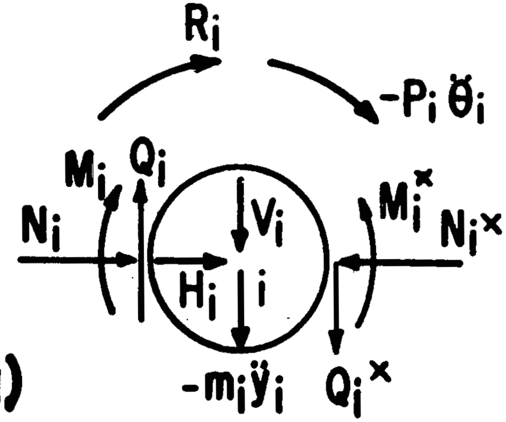
FIGURE 11a: LUMPED MASS SYSTEM



b)

FIGURE 11b: EXTERNAL FORCES ACTING ON THE i^{th} MASS
(FOR THE ORDINARY BEAM CASE $R_i = 0$; $P_i = 0$)

a)



b)

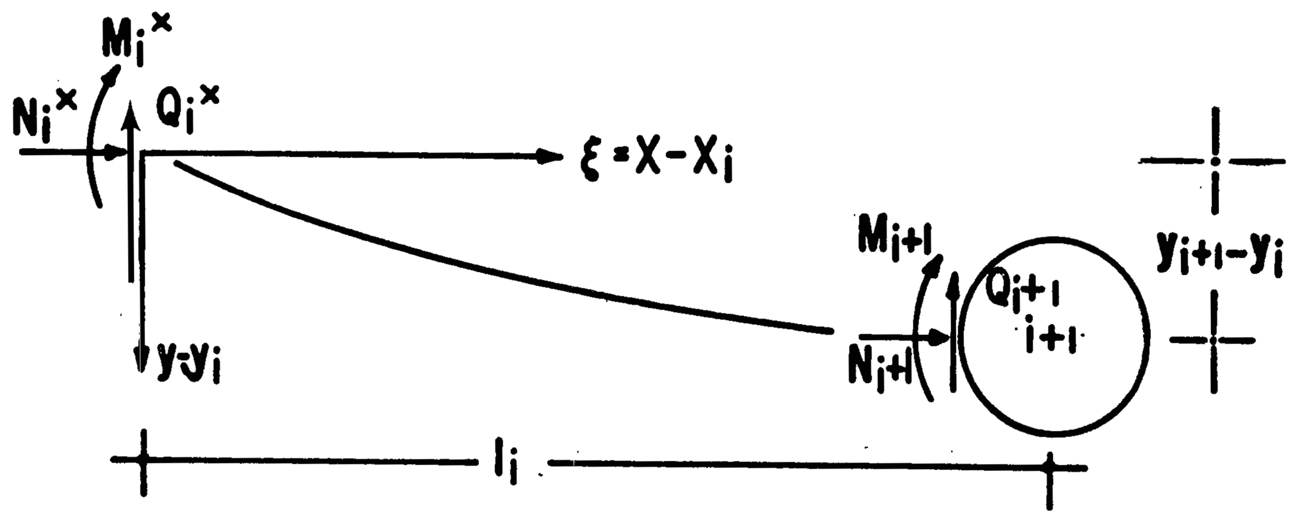


FIGURE 12a: EXTERNAL AND INTERNAL FORCES ACTING ON THE i^{th} MASSES
(FOR THE ORDINARY BEAM CASE $R_i = 0$; $P_i = 0$)

FIGURE 12b: INTERNAL FORCES ACTING ON THE i^{th} SECTION

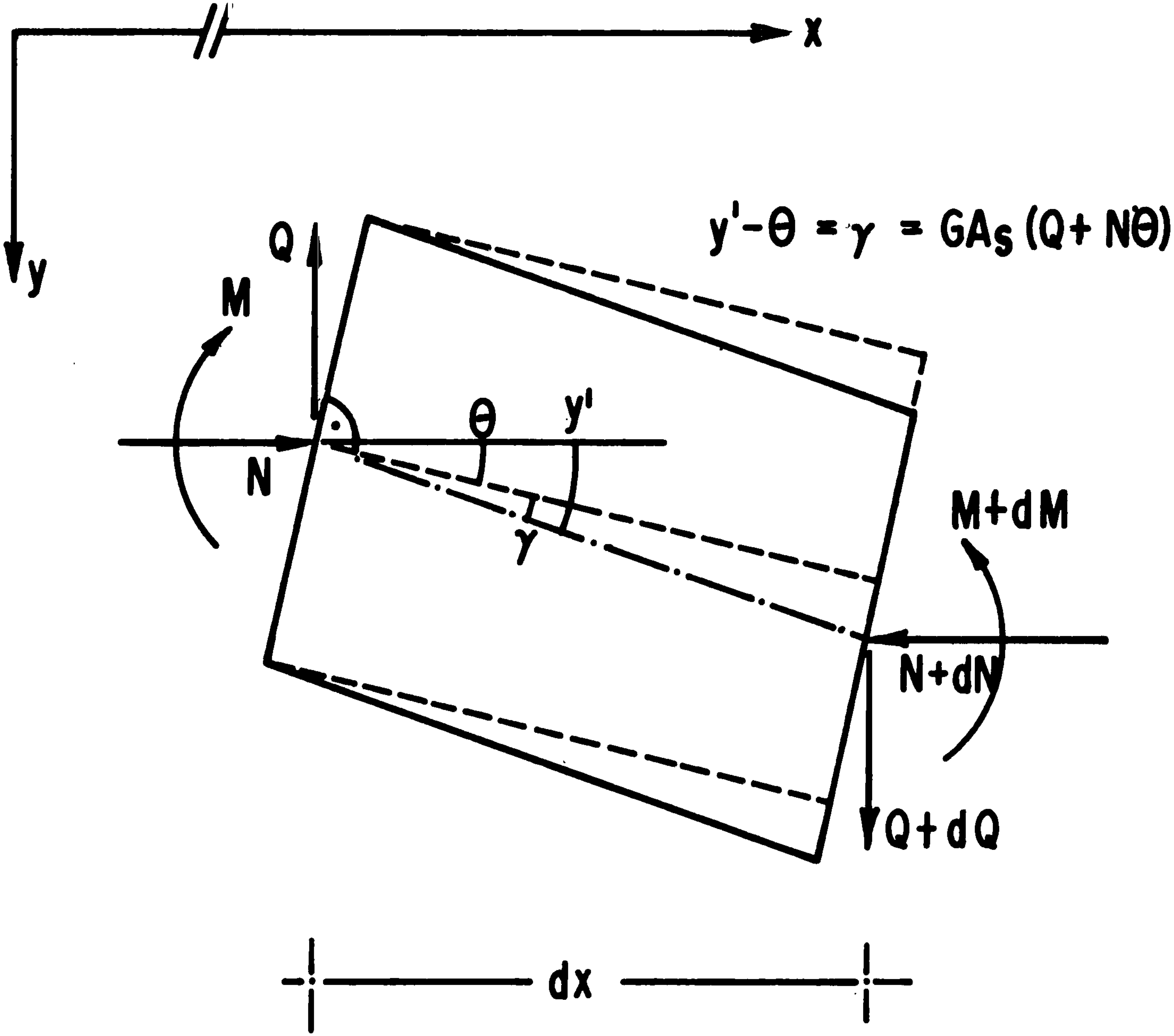


FIGURE 13. SHEAR DEFORMATION OF A BEAM ELEMENT

TABLE 1 SATURN V 3 STAGE LAUNCH VEHICLE; MASS AND STIFFNESS DATA

ST.	l_i IN.	m_i LB.IN.$^{-1}$SEC^2	P_i LB.IN.SEC^2	$I_i \times 10^{-5}$ IN.^{4}	A_{si} IN^2
1		20.537	259		
	89			.10	1.5
2		72.179	1 554		
	170			.30	5.0
3		64.941	4 144		
	181			.84	14.0
4		65.635	24 088		
	156			4.00	36.5
5		14.952	33 412		
	158			6.50	38.5
6		86.618	65 788		
	235			11.50	68.0
7		332.580	140 642		
	213			23.25	86.5
8		24.119	145 045		
	132			36.00	92.0
9		124.549	221 193		
	203.5			50.00	127.5
10		361.208	476 057		
	335.5			50.00	127.5
11		1791.633	468 287		
	184			54.00	137.5
12		345.742	238 806		
	123			54.00	137.5
13		57.606	193 [illegible]38		
	140			52.00	132.5
14		1866.523	3[illegible] 152		
	244.5			69.50	177.5
15		3297.779	591 575		
	244.5			79	203.0
16		2682.730	615 404		
	310			53.00	135.0
17		1938.913	546 767		
	237			65.00	166.0
18		1538.302	340 078		
	85			[illegible]2.00	209.0
19		512.671	376 857		
	180			107.00	273.0
20		341.764	283 873		

TABLE 2 SATURN V 3 STAGE VEHICLE; THE FIRST THREE FREQUENCIES

$\alpha = \bar{\alpha} g$ $\bar{\alpha}$	f_1 f_2 ORDINARY BEAM	f_3 SEC^{-1} TIMOSHENKO BM.
0	1.248	1.170
	2.574	2.213
	4.502	3.484
1	1.245	1.166
	2.568	2.206
	4.495	3.477
2	1.242	1.162
	2.563	2.199
	4.488	3.470
3	1.239	1.158
	2.558	2.192
	4.482	3.463
4	1.236	1.154
	2.553	2.186
	4.476	3.456
5	1.234	1.150
	2.548	2.179
	4.469	3.449
6	1.233	1.146
	2.543	2.172
	4.463	3.441
7	1.228	1.142
	2.537	2.165
	4.457	3.434
8	1.225	1.138
	2.532	2.159
	4.450	3.427

TABLE 3 SATURN V 3 STAGE LAUNCH VEHICLE; DEVIATIONS FROM THE ORTHOGONALITY CONDITIONS OF THE FIRST THREE NORMALIZED MODES

$\alpha = \bar{\alpha} g$	ORDINARY BEAM			TIMOSHENKO BEAM		
$\bar{\alpha}$	1, 2	1, 3	2, 3	1, 2	1, 3	2, 3
0	.000 049	.000 374	-.000 475	-.000 115	.000 317	-.000 411
1	-.002 579	.001 566	-.001 164	-.001 908	.001 083	-.000 821
2	-.005 155	.003 154	-.002 363	-.003 865	.002 169	-.001 639
3	-.007 794	.004 756	-.003 508	-.005 901	.003 335	-.002 539
4	-.010 457	.006 526	-.004 929	-.007 986	.004 505	-.003 436
5	-.013 114	.008 150	-.006 167	-.010 126	.005 692	-.004 344
6	-.015 821	.009 898	-.007 594	-.012 343	.006 962	-.005 333
7	-.018 577	.011 810	-.009 059	-.014 625	.008 252	-.006 344
8	-.021 320	.013 382	-.010 215	-.016 975	.009 563	-.007 349

APPENDIX

REPRESENTATION OF THE INERTIA LOADS REFERRED TO A BODY-FIXED COORDINATE SYSTEM

The loads acting on the launch vehicle during powered flight consist of the thrust and the inertia forces of the vehicle masses. To make proper allowance for these loads it is necessary to represent them in the body-fixed reference system of coordinates. The position of the thrust vector, whose amount is known, is at all times determined by deflection and slope of the vehicle end. To represent the inertia forces, the acceleration of the vehicle masses must be described in the accelerated, moving, body-fixed coordinate system.

In general, the velocity vector of a point referred to a moving coordinate system is given by

$$\bar{v} = \bar{v}_o + \bar{\omega}_\wedge \bar{r} + \frac{d\bar{r}}{dt} \tag{A1}$$

where

$\bar{v}_o$ is the velocity vector of the origin,

$\bar{\omega}$ is the rotation vector of the coordinate system,

$\bar{r}$ is the location vector of the mentioned point, hence $\frac{d\bar{r}}{dt}$ is the relative velocity vector.

The acceleration of the point $\bar{r}$ may be expressed by

$$\bar{a} = \bar{\omega}_\wedge \bar{v} + \frac{d\bar{v}}{dt} \tag{A2}$$

Without loss of generality, it can be assumed that the plane lateral deformation of the vehicle axis coincides at all times with the x, y-plane; hence $\bar{v}_o$ and $\bar{\omega}$ may be expressed in the body-fixed system:

$$\bar{v}_o = \begin{bmatrix} v_{o1} \\ v_{o2} \\ 0 \end{bmatrix} \tag{A3}$$

$$\overline{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \tag{A4}$$

where ω is the angle velocity of the rotating system.

The location vector of any point of the (deformed) vehicle axis is given by:

$$\overline{r} = \begin{bmatrix} x \\ y\ (xt) \\ 0 \end{bmatrix} \tag{A5}$$

Since longitudinal deformations will not be considered, the coordinate x is independent of time.

From equations (A1), (A3), (A4), and (A5) the velocity of the point r can be obtained as:

$$\overline{v} = \begin{bmatrix} v_{01} - \omega\ y \\ v_{02} + \omega\ x + y_t \end{bmatrix} \tag{A6}$$

The third vanishing component is omitted in equation (6) and the following equations. The acceleration of the point $\overline{r}$ may be concluded from equations (A2), (A3), (A4), (A5), and (A6) as

$$\overline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \dot{v}_{01} - \omega\ v_{02} - \omega^2 x - \dot{\omega}\ y - 2\ \omega\ y_t \\ \dot{v}_{02} + \omega\ v_{01} - \omega^2 y + \dot{\omega}\ x + y_{tt} \end{bmatrix} \tag{A7}$$

Since motion relative to the coordinate system does not occur, the center of mass remains at all times on the x-axis having the x-coordinate η, and hence

$$\dot{\eta} = 0$$

Therefore, application of (A7) to the center of mass yields:

$$\bar{a}_\eta = \begin{bmatrix} a_{\eta 1} \\ a_{\eta 2} \end{bmatrix} = \begin{bmatrix} \dot{v}_{01} - \omega\, v_{02} - \omega^2 \eta \\ \dot{v}_{02} + \omega\, v_{01} + \dot{\omega}\, \eta \end{bmatrix} \tag{A8}$$

which represents the acceleration vector of the C. M. Then, application of (A7) and (A8) to the i^{th} lumped-mass of the system shown by Figure 11 yields:

$$\bar{a}_i = \bar{a}_\eta + \begin{bmatrix} 0 \\ \dot{\omega}\,(x_i - \eta) + \ddot{y}_i \end{bmatrix} - \begin{bmatrix} \omega^2 (x_i - \eta) + \dot{\omega}\, y_i + 2\,\omega\, \dot{y}_i \\ \omega^2\, y_i \end{bmatrix} \tag{A9}$$

Considering the third equation (9) one may recognize that the third term of (A9) contains second and third order quantities only. Under the assumption of small oscillations, this term may be neglected. Now, (A9) can be written

$$\bar{a}_i = \begin{bmatrix} a_{\eta 1} \\ a_{\eta 2} \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\omega}\,(x_i - \eta) \end{bmatrix} + \begin{bmatrix} 0 \\ \ddot{y}_i \end{bmatrix} \tag{A10}$$

The three terms represent acceleration of the C. M., acceleration of the i^{th} mass because of the vehicle rotation about its C. M. and the relative acceleration of the i^{th} mass. Consequently, the inertia forces equation (4) acting on the i^{th} vehicle mass

$$H_i\ ,\ V_i,$$

(Fig. 11b and 12a) can be expressed by means of equation (A10) as follows:

$$H_i = -\, m_i\, a_{\eta 1}$$

$$V_i = -\, m_i\, [a_{\eta 2} + \dot{\omega}\,(x_i - \eta)]$$

Q. E. D.

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